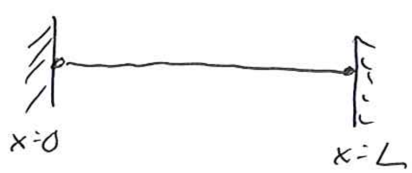
Standing Waves

- Similar to traveling waves, standing waves are <u>everywhere</u>
  - Musical instruments
    - guitar strings
    - drum head
    - air columns in woodwind / brass instruments
  - Microwave oven
  - lasers / optical cavities 
  - quantum wavefunctions $\longleftrightarrow$ standing wave solutions of Schrödinger eq.

//

Standing waves on a string.

- Consider the same taut string we considered in previous lectures

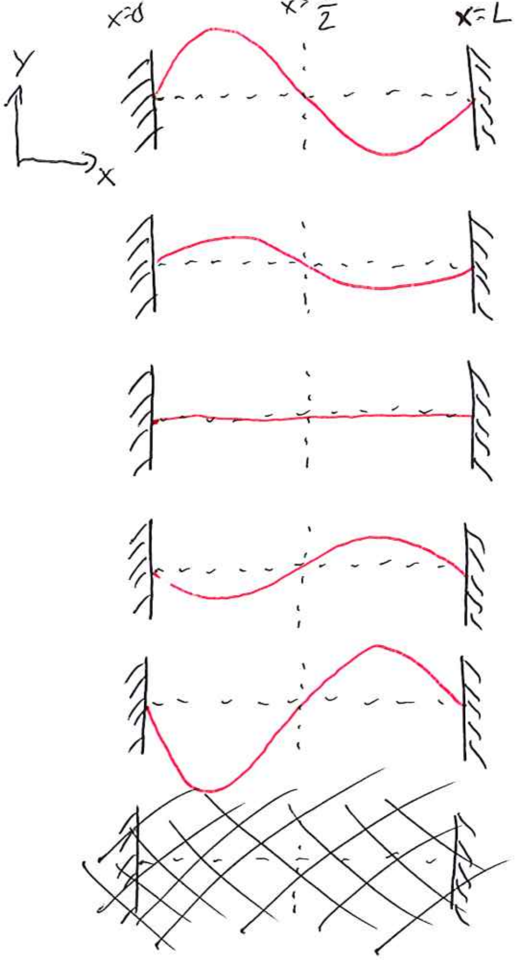


undergoing transverse vibrations

- Because the ends of the string at $x=0, L$ are held fixed, the y-displacement at those points is zero

  → These acts as boundary conditions for the types of standing waves that can form

· Consider an example of a standing wave on this string



snapshots in time of standing waves

· positions where $y=0$ called "nodes"

$\rightarrow x=0, \frac{L}{2}, L$ in this example

· positions where $|y| = \max$ called "antinodes"

$\rightarrow x = \frac{L}{4}, \frac{3L}{4}$ in this example

· * positions of nodes & antinodes <u>do not move</u>

$\rightarrow$ "standing waves" or "stationary waves"

· All elements/particles in standing wave oscillate at the same frequency

· However, amplitude of oscillation at each point x varies

ansatz

$$\longrightarrow \quad y(x,t) = \underbrace{f(x)}_{\text{x-dependent amplitude}} \overbrace{\cos(\omega t + \phi)}^{\text{temporal oscillation}}$$

[intentionally separated]

· If we choose the max amplitude to occur at $t=0$, then $\phi = 0$

$$\rightarrow y(x,t) = f(x)\cos\omega t$$

· We now substitute this equation for $y(x,t)$ into the wave eq: $\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$

$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 f(x)\cos\omega t \qquad \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 f(x)}{\partial x^2}\cos\omega t$$

$$\Rightarrow \quad -\omega^2 f(x)\cancel{\cos\omega t} = v^2 \frac{\partial^2 f(x)}{\partial x^2}\cancel{\cos\omega t}$$

$$\Rightarrow \quad \underbrace{\frac{\partial^2 f(x)}{\partial x^2} = -\frac{\omega^2}{v^2} f(x)}$$

Looks very similar to simple harmonic osc!

$$\frac{d^2 x}{dt^2} = -\omega_0^2 x \quad , \text{ w/ solutions}$$

$$x = A\cos\omega t + B\sin\omega t$$

$\Rightarrow$ This similarity suggest solutions:

$$f(x) = A\sin\left(\frac{\omega}{v}x\right) + B\cos\left(\frac{\omega}{v}x\right)$$

where A, B constants determined by boundary conditions

$\longrightarrow$

- In our example, boundary conditions are:

(1) $f(x=0)=0$, (2) $f(x=L)=0$

- Plug into our ansatz solution:

$$f(x=0) = \underbrace{A\sin\left(\frac{\omega}{v}0\right)}_{=0} + \boxed{B\cos\left(\frac{\omega}{v}0\right) = 0}$$

→ Boundary condition (1) gives that $\underline{\cancel{A}\, B = 0}$

$$f(x=L) = A\sin\left(\frac{\omega}{v}L\right) = 0$$

→ satisfied when $\frac{\omega L}{v} = n\pi$ where $n = 1, 2, 3, \ldots$

→ $\boxed{\omega_n = \frac{n\pi v}{L}}$ Each value of n has a different associated frequency ω_n

$$\Rightarrow f_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$$

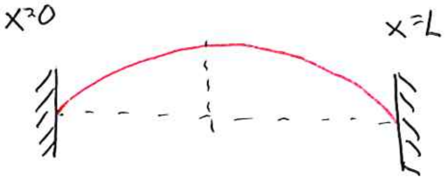
- Plugging into our expression for $y(x,t)$

$$\Rightarrow \boxed{y_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right)\cos\omega_n t}$$

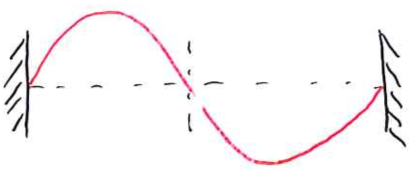
Describes standing waves on a string!

- Each value n in $y_n(x,t)$ corresponds to a different <u>mode</u> of vibration (normal modes)

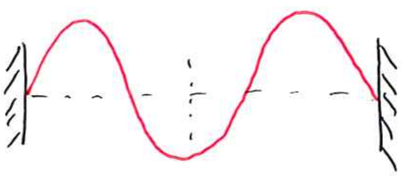
- Consider $n=1 \rightarrow 4$



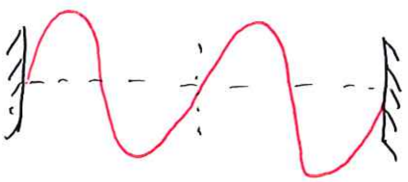
Fundamental (1st harmonic) $n=1$ $\qquad f_1(x) = A_1 \sin\left(\frac{\pi}{L}x\right)$



2nd harmonic $n=2$ $\qquad f_2(x) = A_2 \sin\left(\frac{2\pi}{L}x\right)$



3rd harmonic $n=3$ $\qquad f_3(x) = A_3 \sin\left(\frac{3\pi}{L}x\right)$



4th harmonic $n=4$ $\qquad f_4(x) = A_4 \sin\left(\frac{4\pi}{L}x\right)$

- We see that the integer n describes the number of antinodes in the mode

- Correspond angular frequencies are $\omega_n = \frac{\pi v}{L}, \frac{2\pi v}{L}, \frac{3\pi v}{L}, \frac{4\pi v}{L} \ldots$

- The period of each oscillating mode is then:

$$T = \frac{2\pi}{\omega_n} = \frac{2L}{nv}$$

- Since $v = \nu\lambda$, $\omega = 2\pi\nu$

$$\omega_n = \frac{n\pi v}{L} = \frac{n\pi}{L}\cancel{\nu}\lambda = 2\pi\cancel{\nu}$$

$$\Rightarrow \boxed{\lambda_n = \frac{2L}{n}}$$ where λ_n is wavelength of n^{th} standing wave

- rewrite as $\boxed{L = \frac{n\lambda_n}{2}}$, we see that we can only obtain standing waves only if an integral number of half-wavelengths can fit between the two fixed ends of the string of separation L

- Each standing wave w/ wavelength λ_n has an associated wavevector $k_n = \frac{2\pi}{\lambda_n} = \frac{2\pi}{\frac{2L}{n}} = \boxed{\frac{n\pi}{L} = k_n}$

- We can rewrite $Y_n(x,t)$ as:

$$\boxed{Y_n(x,t) = A_n \sin k_n x \cos \omega_n t}$$ (Alternativ expression)

- Let's examine the fundamental mode $n=1$

$$\omega_1 = \frac{\pi v}{L} \Rightarrow \nu_1 = \frac{\omega_1}{2\pi} = \frac{v}{2L}$$

- We previously learned that $v = \sqrt{\frac{T}{\mu}}$ for a taut string

$$\Rightarrow \nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

→ Shows that frequency of oscillation depends on L, T, μ. Relevant for guitar string

→ 6 guitar strings have same L, similar T. Different μ (thickness) gives different pitch

→ When press a string against a fret, the length L shortens, and pitch increases

- The frequencies ~~from~~ of all of the harmonics together form the "harmonic series", where different individual harmonics n have different amplitudes A_n, depending on details like materials involved, which affects frequency-dependent damping.

→ harmonic series unique to each instrument, which is why every instrument sounds different!

Standing waves as superposition of 2 traveling waves

- In previous lectures we saw that the general solution to the one-dimensional wave eq. is: $y = \underbrace{f(x-vt)}_{+\hat{x} \text{ moving wave}} + \underbrace{g(x+vt)}_{-\hat{x} \text{ moving wave}}$

- For sinusoidal waves, we get solutions like:

$$y = \frac{A}{2}\sin(kx-\omega t) + \frac{A}{2}\sin(kx+\omega t)$$

$\left[k = \frac{2\pi}{\lambda}\right]$
$\omega = kv$

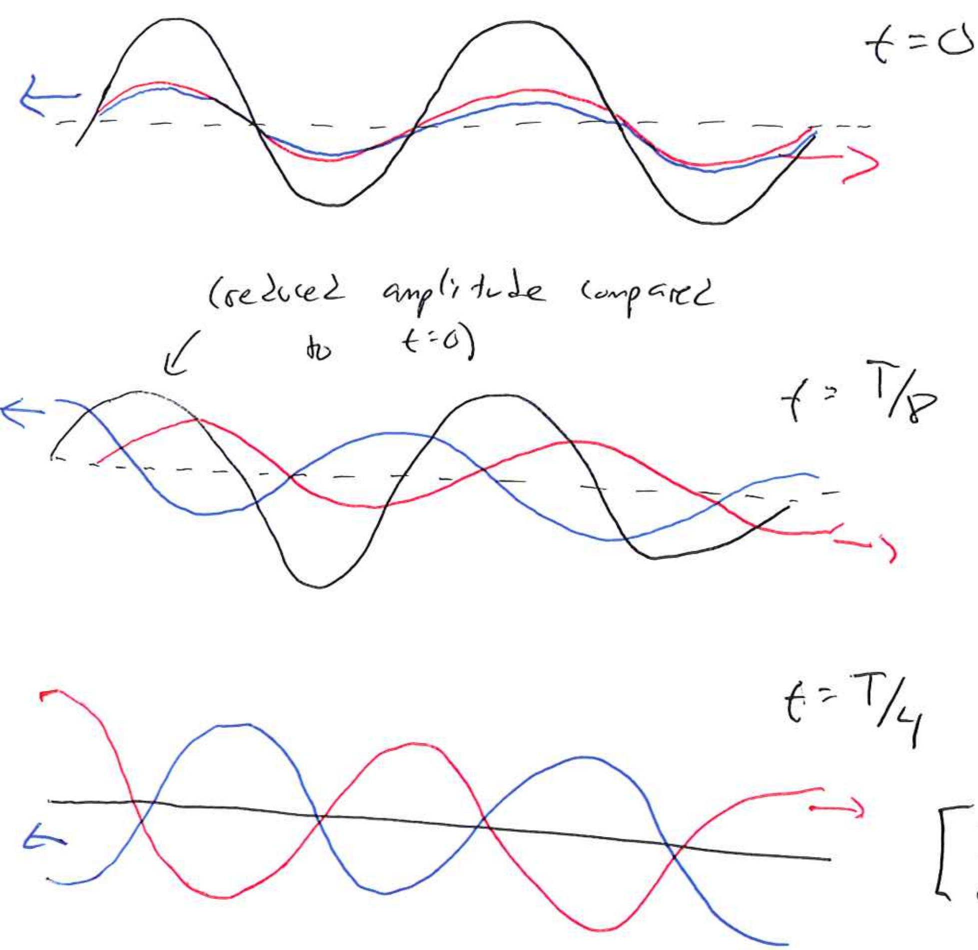
- Use identity: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$

$\Rightarrow \underbrace{y = A\sin kx\cos\omega t}$

↳ This is of identical form to what we just derived for standing waves!

$\Rightarrow$ **Standing wave is a superposition of 2 traveling waves of same frequency & amplitude traveling ~~moving~~ in opposite directions.**

⟶



— Left moving wave

— Right moving wave

— Standing wave (left + right)

[I accidentally switched ~~the~~ which wave ~~colors of~~ should be red/blue here]

- The above sketch is just considering two waves of identical amplitude & frequency moving in opp. directions to give a net wave that is stationary
- In our example, the two rigid walls act as reflection points for traveling waves, which give a π phase shift (ie amplitude reverses direction) upon reflection
- The boundary conditions (ie, length of the string) determine which traveling waves interfere constructively